

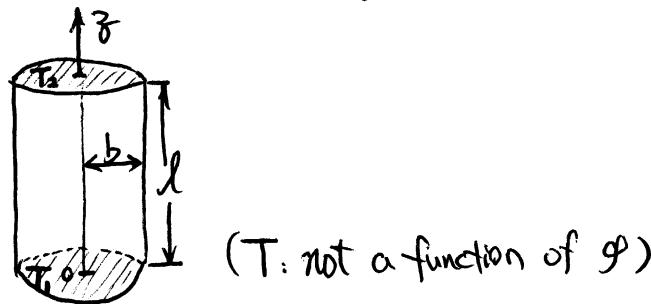
2.3. Separation of Variables - Cylindrical System

Steady-state heat conduction equation:

$$\underbrace{\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{k} g(r, \theta, z) = 0}$$

* Example 1.

Consider a cylinder of radius b and length l . The surface and one base are held at constant temperature T_1 , and the other base at T_2 . There is no heat generation. Find the steady-state temperature distribution $T(r, z)$.



The complete problem:

$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} = 0$ or $\frac{\partial^2 T(r, z)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r, z)}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0$
B.C. $\begin{cases} T _{r=b} = T_1 \\ T _{r=0} = \text{finite} \\ T _{z=0} = T_1 \\ T _{z=l} = T_2 \end{cases}$

Define: $\underline{\theta(r, \varphi)} = T(r, \varphi) - \bar{T}_1$

Therefore: $\frac{\partial^2 \theta(r, \varphi)}{\partial r^2} + \frac{1}{r} \frac{\partial \theta(r, \varphi)}{\partial r} + \frac{\partial^2 \theta(r, \varphi)}{\partial \varphi^2} = 0$

B.C. $\left\{ \begin{array}{l} \theta|_{r=b} = 0 \\ \theta|_{r=0} = \text{finite} \\ \theta|_{\varphi=0} = 0 \\ \theta|_{\varphi=\ell} = \bar{T}_2 - \bar{T}_1 = \theta_\ell \end{array} \right.$

\leftarrow Nonhomogeneous

① Separation of $\theta(r, \varphi)$

Assume: $\theta(r, \varphi) = R(r) Z(\varphi)$

then: $Z'' R'' + \frac{1}{r} Z' R' + R Z'' = 0$

$$\frac{1}{R} (R'' + \frac{1}{r} R') + \frac{1}{Z} Z'' = 0$$

i.e. $\frac{R''(r) + \frac{1}{r} R'(r)}{R} = - \frac{Z''(\varphi)}{Z(\varphi)} = \mu \quad (\text{const.})$

or: $\left\{ \begin{array}{l} R''(r) + \frac{1}{r} R'(r) - \mu R(r) = 0 \\ Z''(\varphi) + \mu Z(\varphi) = 0 \end{array} \right.$

② Solve ODEs.

There are three cases to investigate:

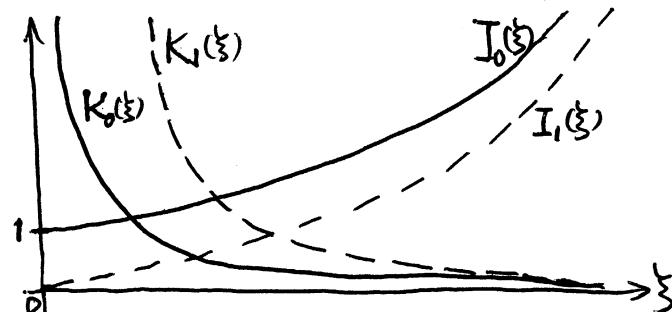
$$\mu > 0, \quad \mu = 0, \quad \mu < 0.$$

(1) For $\mu > 0$ Let $\mu = \lambda^2$

$$\text{Therefore: } R''(r) + \frac{1}{r}R'(r) - \lambda^2 R(r) = 0$$

$$R(r) = A I_0(\lambda r) + B K_0(\lambda r)$$

$I_0(\xi)$ and $K_0(\xi)$ are modified Bessel functions of order 0.



$$\text{Also: } Z''(\xi) + \lambda^2 Z(\xi) = 0$$

$$Z(\xi) = C \cos \lambda \xi + D \sin \lambda \xi$$

imposing B.C.: $\theta|_{r=0} = \text{finite}$.

$$\text{i.e. } R|_{r=0} = \text{finite} \Rightarrow B = 0$$

($K_0(\lambda r)$ diverges at $r=0$)

$$\text{and, } R(r) = A I_0(\lambda r)$$

imposing B.C.: $\theta|_{r=b} = 0$

$$\text{i.e. } R|_{r=b} = 0$$

$$R|_{r=b} = A I_0(\lambda b) = 0 \Rightarrow A = 0$$

($I_0(\lambda r) \neq 0$)

$$\text{and, } R(r) = 0$$

not a meaningful solution!

Conclusion: μ cannot be greater than 0 !

(2) For $\mu = 0$

We have: $\begin{cases} R''(r) + \frac{1}{r}R'(r) = 0 \\ R(r) = A \ln r + B \end{cases}$

and: $\begin{cases} Z''(\theta) = 0 \\ Z(\theta) = C\theta + D \end{cases}$

Imposing B.C. $O|_{r=0} = \text{finite}$

i.e. $R|_{r=0} = \text{finite} \Rightarrow A = 0$

($\ln r \rightarrow -\infty$ as $r \rightarrow 0$)

so, $R(r) = B$

Imposing B.C. $O|_{r=b} = 0$

i.e. $R|_{r=b} = 0 \Rightarrow B = 0$

so: $R(r) = 0$ not a meaningful solution!

Conclusion: μ cannot be 0!

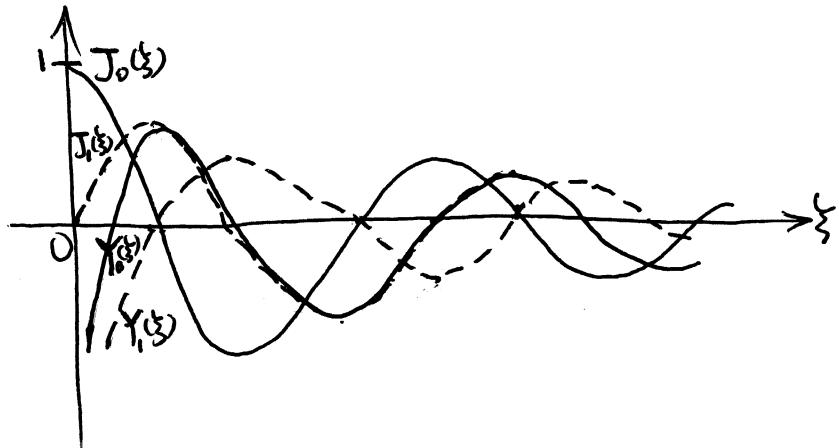
(3) For $\mu < 0$ let $\mu = -\lambda^2$ ($\lambda > 0$)

Therefore: $\begin{cases} R''(r) + \frac{1}{r}R'(r) + \lambda^2 R(r) = 0 \\ R(r) = A J_0(\lambda r) + B Y_0(\lambda r) \end{cases}$

also: $\begin{cases} Z''(\theta) - \lambda^2 Z(\theta) = 0 \end{cases}$

$Z(\theta) = C \cosh \lambda \theta + D \sinh \lambda \theta$

$J_0(\xi)$ and $Y_0(\xi)$ are Bessel functions of order 0.



Impose B.C. $\theta|_{r=0}$ = finite.

i.e. $R|_{r=0}$ = finite $\Rightarrow B = 0$

($Y_0(\lambda r)$ diverges at $r=0$)

so. $R(r) = A J_0(\lambda r)$

Impose B.C. $\theta|_{r=b} = 0$

i.e. $R|_{r=b} = 0$

$$R|_{r=b} = A J_0(\lambda b) = 0$$

If $A=0 \Rightarrow R(r)=0$ not a meaningful solution.

We must have: $\underline{J_0(\lambda b) = 0}$

Therefore: λ can only take certain values, λ_m (eigenvalues)
defined by: $\boxed{J_0(\lambda_m b) = 0}$ $m=1, 2, 3, \dots$

For each λ_m : $\underline{R_m(r) = A_m J_0(\lambda_m r)}$.

Imposing B.C. $\theta|_{\beta=0} = 0$
 i.e., $Z|_{\beta=0} = 0$
 $Z|_{\beta=0} = C \cosh \lambda \beta|_{\beta=0} + D \sinh \lambda \beta|_{\beta=0} = C = 0$
 so, $\underline{Z(\beta)} = D \sinh \lambda \beta$

(3) Making final solution.

for each λ_m : $\theta_m(r, \beta) = A_m J_0(\lambda_m r) \cdot \sinh(\lambda_m \beta)$ ($m=1, 2, 3, \dots$)

With λ_m defined by $[J_0(\lambda_m b) = 0]$

Therefore, $\underline{\theta(r, \beta)} = \sum_{m=1}^{\infty} A_m J_0(\lambda_m r) \sinh(\lambda_m \beta)$

(4) Determine unknown coefficients.

Applying nonhomogeneous B.C. $\theta|_{\beta=l} = \theta_e$

so, $\theta_e = \sum_m A_m J_0(\lambda_m r) \sinh(\lambda_m l)$

Using orthogonal property of Bessel function:

$$\int_0^b J_0(\lambda_m r) J_0(\lambda_n r) r dr = \begin{cases} 0 & (m \neq n) \\ N_m & (m = 0) \end{cases}$$

then:

$$\int_0^b \underline{\theta_e J_0(\lambda_m r) r dr} = \int_0^b \sum_m A_m J_0(\lambda_m r) \sinh(\lambda_m l) \cdot \underline{J_0(\lambda_m r) r dr}$$

i.e., $\theta_e \int_0^b \underline{J_0(\lambda_m r) r dr} = A_m \sinh(\lambda_m l) \underbrace{\int_0^b J_0^2(\lambda_m r) r dr}_{N_m}$

It can be shown:

$$\left\{ \begin{array}{l} \int_0^b J_0(\lambda_m r) r dr = \frac{b}{\lambda_m} J_1(\lambda_m b) \\ N_m = \int_0^b J_0^2(\lambda_m r) r dr = \frac{b^2}{2} J_1^2(\lambda_m b) \end{array} \right.$$

therefore: $\theta_e \cdot \left[\frac{b}{\lambda_m} J_1(\lambda_m b) \right] = A_m \sinh(\lambda_m l) \left[\frac{b^2}{2} J_1^2(\lambda_m b) \right]$

i.e., $A_m = \frac{2\theta_e}{b \lambda_m \sinh(\lambda_m l) J_1(\lambda_m b)}$

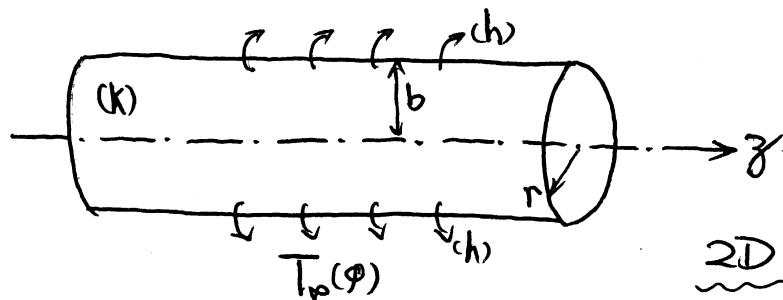
so: $\theta(r, \varphi) = \frac{2\theta_e}{b} \sum_{m=1}^{\infty} \frac{1}{\lambda_m} \frac{\sinh \lambda_m r}{\sinh \lambda_m l} \cdot \frac{J_0(\lambda_m r)}{J_1(\lambda_m b)}$

or: $\overline{T}(r, \varphi) = T_1 + \frac{2(T_2 - T_1)}{b} \sum_{m=1}^{\infty} \frac{1}{\lambda_m} \frac{\sinh \lambda_m r}{\sinh \lambda_m l} \cdot \frac{J_0(\lambda_m r)}{J_1(\lambda_m b)}$

with λ_m defined by: $J_0(\lambda_m b) = 0$.

* Example 2.

Determine Steady-State temperature distribution in a solid cylinder, which is subject to convective heat transfer at boundary surface $r=b$, with ambient temperature varying as $T_{\infty}(\phi)$.



2D problem: $T(r, \phi)$

The complete problem:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} = 0.$$

$$\text{B.C. } \left. T \right|_{r=0} = \text{finite}$$

$$\left. \frac{k^2 T}{\partial r} \right|_{r=b} + h T \Big|_{r=b} = h T_{\infty}(\phi) \quad \leftarrow \text{Nonhomogeneous}$$

$$\text{And: } T(r, \phi) = T(r, \phi + 2\pi)$$

\leftarrow periodic requirement

① Separation of $T(r, \phi)$:

$$\text{Assume: } T(r, \phi) = R(r) \Phi(\phi)$$

$$\text{then: } R'' \Phi + \frac{1}{r} R' \Phi + \frac{1}{r^2} R \Phi'' = 0$$

$$\frac{r^2 R'' + r R'}{R} + \frac{\Phi''}{\Phi} = 0$$

$$\frac{r^2 R'' + r R'}{R} = -\frac{\Phi''}{\Phi} = \mu \quad (\text{not a function of } r \text{ and } \phi)$$

$$\text{So: } \begin{cases} r^2 R'' + r R' - \mu R = 0 \\ \bar{\Phi}'' + \mu \bar{\Phi} = 0 \end{cases}$$

② Solving ODEs.

There are three basic cases for μ : $\begin{cases} \mu < 0 \\ \mu = 0 \\ \mu > 0 \end{cases}$.

(1) For $\mu < 0$ Let $\mu = -\lambda^2$

$$\text{So: } \begin{cases} r^2 R'' + r R' + \lambda^2 R = 0 \\ \bar{\Phi}'' - \lambda^2 \bar{\Phi} = 0 \end{cases}$$

$$\text{Note: } \bar{\Phi}(\varphi) = A e^{\lambda \varphi} + B e^{-\lambda \varphi} \quad (\text{if } \mu < 0)$$

It is not a periodic function, does not satisfy
 $\underbrace{\bar{\Phi}(\varphi)}_{=} = \bar{\Phi}(\varphi + 2\pi)$

So: μ cannot be smaller than 0!

(2) For $\mu = 0$

$$\text{We have: } \begin{cases} r^2 R'' + r R' = 0 \\ R(r) = A \ln r + B \end{cases}$$

$$\text{and: } \begin{cases} \bar{\Phi}'' = 0 \\ \bar{\Phi}(\varphi) = C \varphi + D \end{cases}$$

$$\text{Imposing B.C.: } \bar{\Phi}(\varphi) = \bar{\Phi}(\varphi + 2\pi)$$

$$\text{it requires: } \underline{C = 0}$$

$$\text{i.e., } \underline{\bar{\Phi}(\varphi) = D}.$$

Imposing B.C. $T|_{r=0}$ finite

$$\text{i.e. } R|_{r=0} \text{ finite} \Rightarrow A=0 \quad (\ln r|_{r \rightarrow 0} \rightarrow -\infty)$$

$$\text{so: } R(r) = B$$

$$\text{Therefore: } T(r, \phi) = B \cdot D$$

$$\text{equivalently, } T(r, \phi) = B \quad (\text{a constant})$$

(3) For $\mu > 0$ Let $\mu = \lambda^2$ ($\lambda > 0$)

$$\begin{aligned} \text{so: } & \begin{cases} r^2 R'' + r R' - \lambda^2 R = 0 \\ R(r) = Ar^\lambda + Br^{-\lambda} \end{cases} \\ & \begin{cases} \bar{\Phi}'' + \lambda^2 \bar{\Phi} = 0 \\ \bar{\Phi}(\phi) = C \cos \lambda \phi + D \sin \lambda \phi \end{cases} \end{aligned}$$

Imposing B.C. $T|_{r=0}$ finite

$$\text{i.e. } R|_{r=0} \text{ finite} \Rightarrow B=0 \quad (r^{-\lambda}|_{r \rightarrow 0} \rightarrow \infty)$$

$$\text{so: } R(r) = Ar^\lambda$$

Imposing B.C. $\bar{\Phi}(\phi) = \bar{\Phi}(\phi + 2\pi)$

$$\text{For: } \bar{\Phi}(\phi) = C \cos \lambda \phi + D \sin \lambda \phi$$

it requires. $\lambda = 1, 2, 3, \dots$ (integers)

Therefore,

$$T_\lambda(r, \phi) = r^\lambda (G \cos \lambda \phi + D \sin \lambda \phi) \quad (\lambda = 1, 2, 3, \dots)$$

③ Making final Solution.

$$T(r, \phi) = B + \sum_{\lambda=1}^{\infty} r^\lambda \underbrace{(C_\lambda \cos \lambda \phi + D_\lambda \sin \lambda \phi)}_{\substack{\text{from } \mu=0 \\ \text{from } \mu>0}}$$

Note 1: As $\lambda \rightarrow \infty$, $r^\lambda \rightarrow 0$.

however, the unknown coefficients C_λ and D_λ will make the sum converge to a finite value.

(e.g., $r^\lambda C_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$.)

Note 2: if $\lambda=0$, $r^\lambda \left[(C_0 \cos 0 \phi + D_0 \sin 0 \phi) \right] \Big|_{\lambda=0} = C_0$ (const.)

so the unknown coefficient "B" can be merged into the sum to allow $\lambda=0$ (i.e., $B=C_0$)

Therefore:

$$\boxed{T(r, \phi) = \sum_{\lambda=0}^{\infty} r^\lambda (C_\lambda \cos \lambda \phi + D_\lambda \sin \lambda \phi)}$$

④ Determine Unknown Coefficients.

Applying Nonhomogeneous B.C. $\left. k \frac{\partial T}{\partial r} \right|_{r=b} + h T \Big|_{r=b} = h T_\infty(\phi)$

$$\text{i.e. } \left. \frac{\partial T}{\partial r} \right|_{r=b} + H T \Big|_{r=b} = f(\phi) \quad \text{with: } \begin{cases} H \equiv \frac{h}{k} \\ f(\phi) = \frac{h}{k} T_\infty(\phi) \end{cases}$$

$$\text{so: } \sum_{\lambda=0}^{\infty} b^{\lambda-1} (\lambda + Hb) (C_\lambda \cos \lambda \phi + D_\lambda \sin \lambda \phi) = f(\phi)$$

To determine C_λ , we operate on both side with $\int_0^{\pi} \cos \lambda \phi d\phi$

To determine D_λ , we operate on both side with $\int_0^{\pi} \sin \lambda \phi d\phi$

Therefore, using orthogonal property of the eigenfunctions,

$$b^{\lambda+1}(\lambda+hb)(C_0 \cos \lambda \varphi + D_0 \sin \lambda \varphi) = \frac{1}{\pi} \int_0^{\pi} f(\varphi') \cos [\lambda(\varphi-\varphi')] d\varphi'$$

(Replace $\frac{1}{\pi}$ by $\frac{1}{2\pi}$ for $\lambda=0$)
 (Will not diverge because $r \leq b$)

$$\text{So: } T(r, \varphi) = \frac{b}{\pi} \sum_{\lambda=0}^{\infty} \left(\frac{r}{b} \right)^{\lambda} \frac{1}{\lambda+hb} \int_0^{\pi} f(\varphi') \cos [\lambda(\varphi-\varphi')] d\varphi'$$

$$\text{Considering: } \begin{cases} h = \frac{b}{k} \\ f(\varphi) = \frac{h}{k} T_{rk}(\varphi) \end{cases}$$

$$\text{We have: } T(r, \varphi) = \underbrace{\frac{b}{\pi} \sum_{\lambda=0}^{\infty} \left(\frac{r}{b} \right)^{\lambda} \frac{h}{\lambda k + hb} \int_0^{\pi} T_{rk}(\varphi') \cos [\lambda(\varphi-\varphi')] d\varphi'}_{(\text{replace } \frac{b}{\pi} \text{ by } \frac{b}{2\pi} \text{ for } \lambda=0)}$$

Note: Fourier Transformation

$$\text{For: } f(\varphi) = \sum_{\lambda=0}^{\infty} (A_{\lambda} \sin \lambda \varphi + B_{\lambda} \cos \lambda \varphi)$$

$$\left\{ \begin{array}{l} A_{\lambda} = \frac{1}{\pi} \int_0^{\pi} f(\varphi) \sin \lambda \varphi d\varphi \quad (\lambda = 0, 1, 2, 3, \dots) \\ B_{\lambda} = \begin{cases} \frac{1}{\pi} \int_0^{\pi} f(\varphi) \cos \lambda \varphi d\varphi & (\lambda = 1, 2, 3, \dots) \\ \frac{1}{\pi} \int_0^{\pi} f(\varphi) d\varphi & (\lambda = 0) \end{cases} \end{array} \right.$$